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L_p-estimates for a viscous compressible fluid in an infinite time interval (model problems)

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Abstract. The paper contains analysis of model problems arising in the proof of maximal regularity L_p-estimates for linearized free boundary problem governing the motion of two viscous compressible fluids. The proof is based on convenient representation formulas for the Fourier-Laplace transformation of solutions of model problems and on the Marcinkiewicz-Mikhlin-Lizorkin theorem .

1 Construction of solutions of model problems.

The motion of two viscous compressible fluids contained in a bounded vessel and separated by a free interface is governed by the evolution free boundary problem

$$\begin{cases} \rho(\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v})-\nabla\cdot T(\boldsymbol{v})+\nabla p(\rho)=0,, \\ \rho_t+\nabla\cdot\rho\boldsymbol{v}=0, \quad x\in\Omega_t^+\cup\Omega_t^-, \quad t>0, \\ [\boldsymbol{v}]=0, \quad [-p(\rho)\boldsymbol{n}+T(\boldsymbol{v})\boldsymbol{n}]=0, \quad V_n=\boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \\ \boldsymbol{v}(x,t)=0, \quad x\in S, \\ \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \quad \theta(x,0)=\theta_0(x), \quad x\in\Omega_0^+\cup\Omega_0^-, \end{cases} \tag{1.1}$$

where $\boldsymbol{v}(x,t)=\boldsymbol{v}^\pm(x,t)$, $\rho(x,t)=\rho^\pm(x,t)$ for $x\in\Omega_t^\pm$, Ω_t^+ and Ω_t^- are bounded domains occupied by the fluids and separated by a free interface $\Gamma_t=\partial\Omega_t^+$. It is given for $t=0$ and should be found for $t>0$. The domain $\Omega=\Omega_t^+\cup\Gamma_t\cup\Omega_t^-$ is fixed; the surface $S=\partial\Omega$ is bounded away from Γ_t. By $T(\boldsymbol{v})\equiv T^\pm(\boldsymbol{v})$ we mean the viscous part of the stress tensor:

$$T^\pm(\boldsymbol{v})=\mu^\pm S(\boldsymbol{v}^\pm)+\mu_1^\pm I\nabla\cdot\boldsymbol{v}^\pm, \quad x\in\Omega_t^\pm,$$

$S(\boldsymbol{v})=(\nabla\boldsymbol{v})+(\nabla\boldsymbol{v})^T$ is the doubled rate-of-strain tensor, $\mu=\mu^\pm,\mu_1=\mu_1^\pm$ are positive constants, $p^\pm(\rho^\pm)$ are positive strictly increasing functions of positive arguments, $\boldsymbol{n}$ is the normal to Γ_t exterior with respect to Ω^+, V_n is the velocity of evolution of Γ_t in the direction $\boldsymbol{n}$, $[u]=u^+-u^-$ is the jump of the function u on Γ_t.

By the rest state we mean the solution of (1.1) with $\boldsymbol{v}^\pm(x,t)=0$. Then the domains $\Omega_t^\pm$ are independent of t and $\rho^\pm(x,t)=const=\overline{\rho}^\pm=M^\pm/|\Omega_\pm|$, where $M^\pm$ are total masses of the fluids and $|\Omega^\pm|=mes\Omega^\pm$. The jump conditions on Γ_t reduce to

$$p^+(\overline{\rho}^+)=p^-(\overline{\rho}^-).$$

The rest state is stable, if solutions of (1.1) with small $\boldsymbol{v}_0^\pm$, $\rho_0^\pm$ close to $\overline{\rho}^\pm$, $\Omega_0^\pm$ close to $\Omega^\pm$ are defined for $t>0$ and tend to $(\boldsymbol{v}=0,\rho=\overline{\rho}^\pm,\Omega^\pm)$ as $t\to\infty$. In the paper [1] this was proved under the assumptions

$$\overline{\rho}^+ = \overline{\rho}^- = \overline{\rho} = M/|\Omega|, \quad M = M^+ + M^-,$$

By passing to the Lagrangian coordinates $\{y\}$ and linearizing (1.1) about the rest state we arrive at the linear problem

$$\begin{cases} \bar{\rho}\boldsymbol{v}_t - \nabla \cdot T(\boldsymbol{v}) + p'(\bar{\rho})\nabla\theta = \boldsymbol{f}(y,t), \\ \theta_t + \bar{\rho}\nabla \cdot \boldsymbol{v} = h(y,t), \quad y \in \Omega_0^+ \cup \Omega_0^-, \quad t > 0, \\ [\boldsymbol{v}] = 0, \quad [-p'(\bar{\rho})\theta\boldsymbol{n} + T(\boldsymbol{v})\boldsymbol{n}] = \boldsymbol{b}(y,t), \quad y \in \Gamma_0, \\ \boldsymbol{v}^-(y,t) = 0, \quad y \in S, \\ \boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad \theta(y,0) = \theta_0(y), \quad y \in \Omega_0^+ \cup \Omega_0^-, \end{cases} \tag{1.2}$$

where $\boldsymbol{f}$, h, $\boldsymbol{b}$, $\boldsymbol{v}_0$, θ_0 are some given functions. In the present paper we study model problems arising in the analysis of (1.2), in particular, we obtain maximal regularity weighted L_p-estimates of solutions of these problems. In combination with the Schauder localization method, our results enable us to estimate the higher order Sobolev norms of the solution of (1.2) by the data and by the weighted L_p- norms of $\boldsymbol{v}$ and θ.

Main attention is given to the problem

$$\begin{cases} \overline{\rho}^\pm \boldsymbol{w}_t^\pm - \nabla \cdot T^\pm(\boldsymbol{w}^\pm) + p^{\pm\prime}(\overline{\rho}^\pm)\nabla\vartheta^\pm = \boldsymbol{f}^\pm, \\ \vartheta_t^\pm + \overline{\rho}^\pm \nabla \cdot \boldsymbol{w}^\pm = h^\pm, \quad \pm y_3 > 0, \quad t > 0, \\ [\boldsymbol{w}] = 0, \quad [T_{\alpha 3}(\boldsymbol{w})] = b_\alpha, \quad \alpha = 1,2, \\ [-p'(\overline{\rho})\vartheta + T_{33}(\boldsymbol{w})] = b_3, \quad y_3 = 0, \\ \boldsymbol{w}^\pm(y,0) = 0, \quad \vartheta^\pm(y,0) = 0, \quad \pm y_3 > 0, \end{cases} \tag{1.3}$$

related to the estimates of the solution of (1.2) near the interface Γ_0.

Having in mind the application of the localization method, we assume that supp$\boldsymbol{w}$ and suppϑ are contained in the closure of the domain $\mathfrak{Q} = \mathfrak{Q}^+ \cup \mathfrak{Q}^-$, where $\mathfrak{Q}^\pm = \mathfrak{Q}' \times I^\pm$, $\mathfrak{Q}' = \{|x_j| \leq d_0\}$, $j = 1,2$, $I^\pm = \{\pm x_3 \in (0, d_0)\}$. Moreover, we assume for a while that $\boldsymbol{f} = 0$ and $h = 0$.

We expand $b_j(y', t)$ in the Fourier series in $\mathfrak{Q}'$:

$$b_j(y',t) = \frac{1}{(2d_0)^2} \sum_{k \in \mathbb{Z}^2} \widehat{b}_j(\xi,t)e^{i\xi\cdot y'}, \quad \xi = (\frac{\pi}{d_0}k_1, \frac{\pi}{d_0}k_2),$$

where $\widehat{b}_j(\xi,t) = \int_{\mathfrak{Q}'} e^{-i\xi\cdot x'} b_j(y',t)dy'$ and make the Fourier-Laplace transform

$$\begin{aligned} \widetilde{\boldsymbol{w}}^\pm(\xi,s,y_3) &= \int_0^\infty e^{-st}dt \int_{\mathfrak{Q}'} \boldsymbol{w}^\pm(y,t)e^{-i\xi\cdot y'}dy' = \mathcal{F}\boldsymbol{w}, \\ \widetilde{\vartheta}^\pm(\xi,s,y_3) &= \int_0^\infty e^{-st}dt \int_{\mathfrak{Q}'} \vartheta^\pm(y,t)e^{-i\xi\cdot y'}dy' = \mathcal{F}\vartheta. \end{aligned} \tag{1.4}$$

This converts problem (1.3) into

$$
\begin{cases}
\overline{\rho} s\widetilde{w}_\alpha - \big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{T}_{\beta\alpha}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}\widetilde{T}_{3\alpha}(\widetilde{\boldsymbol{w}})\big) + p'(\overline{\rho}) i\xi_\alpha \widetilde{\vartheta} = 0, \quad \alpha = 1,2, \\
\overline{\rho} s\widetilde{w}_3 - \big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{T}_{\beta 3}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}\widetilde{T}_{33}(\widetilde{\boldsymbol{w}})\big) + p'(\overline{\rho}) \dfrac{d}{dy_3} \widetilde{\vartheta} = 0, \\
s\widetilde{\vartheta} + \overline{\rho}\big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{w}_\beta + \dfrac{d}{dy_3}\widetilde{w}_3\big) = 0, \quad \pm y_3 > 0, \\
[\widetilde{\boldsymbol{w}}] = 0, \quad [\widetilde{T}_{3\alpha}(\widetilde{\boldsymbol{w}})] = \widetilde{b}_\alpha, \quad \alpha = 1,2, \\
[-p'(\overline{\rho})\widetilde{\vartheta} + \widetilde{T}_{33}(\widetilde{\boldsymbol{w}})] = \widetilde{b}_3, \quad y_3 = 0, \\
\widetilde{\boldsymbol{w}}, \widetilde{\vartheta} \to 0, \quad |y_3| \to \infty,
\end{cases}
\tag{1.5}
$$

where $\widetilde{T}^\pm(\widetilde{\boldsymbol{w}}^\pm) = \mathcal{F}T^\pm(\boldsymbol{w}^\pm)$, $\widetilde{S}^\pm(\widetilde{\boldsymbol{w}}^\pm) = \mathcal{F}S^\pm(\boldsymbol{w}^\pm)$, $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}^\pm$, $\widetilde{\vartheta} = \widetilde{\vartheta}^\pm$ for $\pm y_3 > 0$.

Along with (1.3), (1.5), we study similar problems for the Lamé system:

$$
\begin{cases}
\overline{\rho}^\pm \boldsymbol{w}_t^\pm - \nabla \cdot T^\pm(\boldsymbol{w}^\pm) = \boldsymbol{f}^\pm, \quad \pm y_3 > 0, \\
[\boldsymbol{w}] = 0, \quad [T_{i3}(\boldsymbol{w})] = b_i, \quad i = 1,2,3, \quad y_3 = 0, \\
\boldsymbol{w}^\pm(y,0) = 0, \quad \pm y_3 > 0.
\end{cases}
\tag{1.6}
$$

and

$$
\begin{cases}
\overline{\rho} s\widetilde{w}_\alpha - \big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{T}_{\beta\alpha}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}\widetilde{T}_{3\alpha}(\widetilde{\boldsymbol{w}})\big) = 0, \quad \alpha = 1,2, \\
\overline{\rho} s\widetilde{w}_3 - \big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{T}_{\beta 3}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}\widetilde{T}_{33}(\widetilde{\boldsymbol{w}})\big) = 0, \quad \pm y_3 > 0, \\
[\widetilde{\boldsymbol{w}}] = 0, \quad [\widetilde{T}_{3i}(\widetilde{\boldsymbol{w}})] = \widetilde{b}_i, \quad i = 1,2,3, \\
\widetilde{\boldsymbol{w}}, \widetilde{\vartheta} \to 0, \quad |y_3| \to \infty.
\end{cases}
\tag{1.7}
$$

We proceed with constructing the solution of problem (1.7) in the explicit form, assuming for a while that $|k| \geq 1$, $|\xi| \geq \pi/d_0$. We follow the same scheme as in [2,3], see also [4]. The general form of the solution of (1.7) is

$$
\begin{aligned}
\widetilde{\boldsymbol{w}}^+(\xi, s, y_3) &= C_1^+ \begin{pmatrix} r^+ \\ 0 \\ i\xi_1 \end{pmatrix} e^{-r^+ y_3} + C_2^+ \begin{pmatrix} 0 \\ r^+ \\ i\xi_2 \end{pmatrix} e^{-r^+ y_3} + C_3^+ \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1^+ \end{pmatrix} e^{-r_1^+ y_3}, \quad y_3 > 0, \\
\widetilde{\boldsymbol{w}}^-(\xi, s, y_3) &= C_1^- \begin{pmatrix} -r^- \\ 0 \\ i\xi_1 \end{pmatrix} e^{r^- y_3} + C_2^- \begin{pmatrix} 0 \\ -r^- \\ i\xi_2 \end{pmatrix} e^{r^- y_3} + C_3^- \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ r_1^- \end{pmatrix} e^{r_1^- y_3}, \quad y_3 < 0,
\end{aligned}
\tag{1.8}
$$

where $r^\pm = \sqrt{\frac{s}{\nu^\pm} + |\xi|^2}$, $r_1^\pm = \sqrt{\frac{s}{2\nu^\pm + \nu_1^\pm} + |\xi|^2}$, $\nu^\pm = \mu^\pm/\overline{\rho}^\pm$, $\nu_1^\pm = \mu_1^\pm/\overline{\rho}^\pm$, i.e.,

$$
\widetilde{\boldsymbol{w}}^+(\xi, s, y_3) = \begin{pmatrix} C_1^+ r^+ + i\xi_1 C_3^+ \\ C_2^+ r^+ + i\xi_2 C_3^+ \\ \mathbb{C}^+ - r_1^+ C_3^+ \end{pmatrix} e^{-r^+ y_3} + C_3^+ \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1^+ \end{pmatrix} (e^{-r_1^+ y_3} - e^{-r^+ y_3}), \quad y_3 > 0, \tag{1.9}
$$

$$\widetilde{\boldsymbol{w}}^-(\xi, s, y_3) = \begin{pmatrix} -C_1^- r^- + i\xi_1 C_3^- \\ -C_2^- r^- + i\xi_2 C_3^- \\ \mathbb{C}^- + r_1^- C_3^- \end{pmatrix} e^{r^- y_3} + C_3^- \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ r_1^- \end{pmatrix} (e^{r_1^+ y_3} - e^{r^- y_3}), \quad y_3 < 0, \tag{1.10}$$

where $\mathbb{C}^\pm = i\xi_1 C_1^\pm + i\xi_2 C_2^\pm$. Substituting (1.9) and (1.10) into the jump conditions in (1.7), we obtain

$$\begin{aligned}
&\mu^+(-r^{+2} C_\alpha^+ - i\xi_\alpha r_1^+ C_3^+ + i\xi_\alpha(\mathbb{C}^+ - r_1^+ C_3^+)) \\
&\quad - \mu^-(-r^{-2} C_\alpha^- + i\xi_\alpha r_1^- C_3^- + i\xi_\alpha(\mathbb{C}^- + r_1^- C_3^-)) = \widetilde{b}_\alpha, \quad \alpha = 1,2, \\
&(2\mu^+ + \mu_1^+)(-r^+ \mathbb{C}^+ + r_1^{2+} C_3^+) - (2\mu^- + \mu_1^-)(r^- \mathbb{C}^- + r_1^{2-} C_3^-) \\
&\quad + \mu_1^+(\mathbb{C}^+ r^+ - |\xi|^2 C_3^+) - \mu_1^-(-\mathbb{C}^- r^- - |\xi|^2 C_3^-) = \widetilde{b}_3, \\
&C_\alpha^+ r^+ + i\xi_\alpha C_3^+ = -C_\alpha^- r^- + i\xi_\alpha C_3^-, \quad \alpha = 1,2, \\
&\mathbb{C}^+ - r_1^+ C_3^+ = \mathbb{C}^- + r_1^- C_3^-,
\end{aligned} \tag{1.11}$$

and, as a consequence,

$$\begin{aligned}
&\mathbb{C}^+ r^+ - |\xi|^2 C_3^+ = -\mathbb{C}^- r^- - |\xi|^2 C_3^-, \\
&\mu^+(-r^{+2} \mathbb{C}^+ + |\xi|^2 r_1^+ C_3^+) - \mu^-(-r^{2-} \mathbb{C}^- - r_1^- |\xi|^2 C_3^-) \\
&\quad - (\mu^+ - \mu^-)|\xi|^2(\mathbb{C}^+ - r_1^+ C_3^+)) = \sum_{\beta=1,2} i\xi_\beta \widetilde{b}_\beta \equiv \widetilde{B}.
\end{aligned} \tag{1.12}$$

At first we find $C_3^\pm$. We replace $\mathbb{C}^+$ with $(\mathbb{C}^+ - r_1^+ C_3^+) + r_1^+ C_3^+$, $\mathbb{C}^-$ with $(\mathbb{C}^- + r_1^- C_3^-) - r_1^- C_3^-$ in (1.11), (1.12) and make use of the formulas

$$\begin{aligned}
&r^{\pm 2} - |\xi|^2 = \frac{s}{\nu^\pm} = \frac{\bar\rho^\pm s}{\mu^\pm}, \quad r_1^{\pm 2} - |\xi|^2 = \frac{\bar\rho^\pm s}{2\mu^\pm + \mu_1}, \\
&r_1^\pm r^\pm - |\xi|^2 = r^\pm(r_1^\pm - r^\pm) + \frac{s}{\nu^\pm} = \frac{r^\pm}{r_1^\pm + r^\pm}\Big(\frac{s}{2\nu^\pm + \nu_1^\pm} - \frac{s}{\nu^\pm}\Big) + \frac{s}{\nu^\pm} = \frac{s}{\nu^\pm} R^\pm = \frac{\bar\rho^\pm s}{\mu^\pm} R^\pm,
\end{aligned}$$

where

$$R^\pm = \frac{r_1^\pm + \varkappa^\pm r^\pm}{r_1^\pm + r^\pm}, \quad \varkappa^\pm = \frac{\nu^\pm}{2\nu^\pm + \nu_1^\pm}.$$

This leads to the system

$$\begin{aligned}
&A_1(\mathbb{C}^+ - r_1^+ C_3^+) - r_1^+ \bar\rho^+ s C_3^+ - r_1^- \bar\rho^- s C_3^- = \widetilde{B}, \\
&A_2(\mathbb{C}^+ - r_1^+ C_3^+) + \bar\rho^+ s(1 - 2R^+) C_3^+ - \bar\rho^- s(1 - 2R^-) C_3^- = \widetilde{b}_3, \\
&A_3(\mathbb{C}^+ - r_1^+ C_3^+) + \bar\rho^+ s \frac{R^+}{\mu^+} C_3^+ - \bar\rho^- s \frac{R^-}{\mu^-} C_3^- = 0,
\end{aligned} \tag{1.13}$$

where

$$A_1 = -\mu^+(r^{+2} + |\xi|^2) + \mu^-(r^{-2} + |\xi|^2), \quad A_2 = -2\mu^+ r^+ - 2\mu^- r^-, \quad A_3 = r^+ + r^-.$$

It follows that $(\bar\rho^+ s C_3^+, \bar\rho^- s C_3^-)$ satisfy

$$\mathcal{L}(\bar\rho^+ s C_3^+, \bar\rho^- s C_3^-)^T = (\widetilde{B}, \widetilde{b}_3)^T \tag{1.14}$$

with

$$\mathcal{L} = \begin{pmatrix} -\frac{A_1}{A_3}\frac{R^+}{\mu^+} - r_1^+ & \frac{A_1}{A_3}\frac{R^-}{\mu^-} - r_1^- \\ -\frac{A_2}{A_3}\frac{R^+}{\mu^+} + 1 - 2R^+ & \frac{A_2}{A_3}\frac{R^-}{\mu^-} - (1-2R^-) \end{pmatrix}.$$

Hence

$$\begin{aligned} C_3^+ &= \frac{1}{\bar{\rho}^+ sD_0}[(A_2\frac{R^-}{\mu^-} - A_3(1-2R^-))\widetilde{B} - (A_1\frac{R^-}{\mu^-} - A_3 r_1^-)\widetilde{b}_3], \\ C_3^- &= \frac{1}{\bar{\rho}^- sD_0}[A_2\frac{R^+}{\mu^+} - A_3(1-2R^+)\widetilde{B} - (A_1\frac{R^+}{\mu^+} + A_3 r_1^+)\widetilde{b}_3], \end{aligned} \tag{1.15}$$

where

$$\begin{aligned} D_0 = A_3 det\mathcal{L} &= (-\mu^+(r^{2+} + |\xi|^2) + \mu^-(r^{2-} + |\xi|^2))(-\frac{R^-}{\mu^-}(1-2R^+) + \frac{R^+}{\mu^+}(1-2R^-)) \\ &+ 2(\mu^+ r^+ + \mu^- r^-)(\frac{r_1^+ R^-}{\mu^-} + \frac{r_1^- R^+}{\mu^+}) + (r^+ + r^-)(r_1^+(1-2R^-) + r_1^-(1-2R^+)). \end{aligned} \tag{1.16}$$

Next, we find $\widetilde{\omega}_\alpha = C_\alpha^+ r^+ + i\xi_\alpha C_3^+ = -C_\alpha^- r^- + i\xi_\alpha C_3^-$ and $\widetilde{\omega}_3 = \mathbb{C}^+ - r_1^+ C_3^+$. By (1.11),

$$\begin{aligned} &-\mu^+ r^+(C_\alpha^+ r^+ + i\xi_\alpha C_3^+) - \mu^- r^-(-C_\alpha^- r^- + i\xi_\alpha C_3^-) \\ &+\mu^+(r^+ - r_1^+)i\xi_\alpha C_3^+ + \mu^-(r^- - r_1^-)i\xi_\alpha C_3^- + i\xi_\alpha(\mu^+ - \mu^-)(\mathbb{C}^+ - r_1^+ C_3^+) = \widetilde{b}_\alpha, \end{aligned}$$

which implies

$$\begin{aligned} \omega_\alpha &= -\frac{\widetilde{b}_\alpha}{\mu^+ r^+ + \mu^- r^-} + \frac{i\xi_\alpha}{\mu^+ r^+ + \mu^- r^-}(\mu^+ C_3^+(r^+ - r_1^+) + \mu^- C_3^-(r^- - r_1^-)) \\ &+ \frac{i\xi_\alpha(\mu^+ - \mu^-)\widetilde{\omega}_3}{\mu^+ r^+ + \mu^- r^-}, \\ \widetilde{\omega}_3 &= -\frac{2\nu^+ + \nu_1^+}{\nu^+ + \nu_1^+}\frac{r_1^+ + r^+}{r^+ + r^-}R^+ C_3^+(r^+ - r_1^+) + \frac{2\nu^- + \nu_1^-}{\nu^- + \nu_1^-}\frac{r_1^- + r^-}{r^+ + r^-}R^- C_3^-(r^- - r_1^-). \end{aligned} \tag{1.17}$$

Hence (1.9), (1.10) can be written in the form

$$\begin{aligned} \widetilde{\boldsymbol{v}}^+ &= \widetilde{\boldsymbol{\omega}} e^{-r^+ y_3} + C_3^+(r_1^+ - r^+)(i\xi_1, i\xi_2, -r_1^+)^T e_1^+(y_3), \quad y_3 > 0, \\ \widetilde{\boldsymbol{v}}^- &= \widetilde{\boldsymbol{\omega}} e^{r^- y_3} + C_3^-(r_1^- - r^-)(i\xi_1, i\xi_2, r_1^-)^T e_1^-(y_3), \quad y_3 < 0, \end{aligned} \tag{1.18}$$

with $\widetilde{\omega}_i$ given in (1.17) and

$$\begin{aligned} C_3^+(r_1^+ - r^+) &= -\frac{1}{r^+ + r_1^+}\frac{\mu^+ + \mu_1^+}{2\mu^+ + \mu_1^+}\frac{1}{\mu^+ D_0}((A_2\frac{R^-}{\mu^-} - A_3(1-2R^-))\widetilde{B} - (A_1\frac{R^-}{\mu^-} - A_3 r_1^-)\widetilde{b}_3), \\ C_3^-(r_1^- - r^-) &= -\frac{1}{r^- + r_1^-}\frac{\mu^- + \mu_1^-}{2\mu^- + \mu_1^-}\frac{1}{\mu^- D_0}((A_2\frac{R^+}{\mu^+} - A_3(1-2R^+))\widetilde{B} - (A_1\frac{R^+}{\mu^+} - A_3 r_1^+)\widetilde{b}_3), \end{aligned} \tag{1.19}$$

The solution of problem (1.7) defined by (1.18) is unique in the class of functions decaying to zero exponentially, as $|y_3| \to \infty$. Indeed, it is easily seen that the solution of a homogeneous problem satisfies the energy relation

$$s\|\widetilde{\boldsymbol{w}}\|^2 + \frac{1}{2}\|\sqrt{\mu}\widetilde{S}(\widetilde{\boldsymbol{w}})\|^2 + \|\sqrt{\mu_1}(\sum_{\gamma=1}^{2} i\xi_\gamma \widetilde{w}_\gamma + \frac{d\widetilde{w}_3}{dy_3})\|^2 = 0, \tag{1.20}$$

where $\|\cdot\|$ is the norm in $L_2(\mathbb{R})$, $\mu = \mu^{\pm}$, $\mu_1 = \mu_1^{\pm}$, $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}^{\pm}$ for $\pm y_3 > 0$. By the analog of the Korn inequality

$$|\xi|^2\|\widetilde{\boldsymbol{w}}\|^2 + \|\frac{d\widetilde{\boldsymbol{w}}}{dx_3}\|^2 \le c\|\widetilde{S}(\widetilde{\boldsymbol{w}})\|^2, \tag{1.21}$$

we have

$$\mathrm{Re}s\|\widetilde{\boldsymbol{w}}\|^2 + c(|\xi|^2\|\widetilde{\boldsymbol{w}}\|^2 + \|\frac{d\widetilde{\boldsymbol{w}}}{dy_3}\|^2) \le 0,$$

which implies $\widetilde{\boldsymbol{w}} = 0$, if $\mathrm{Re}s \ge -c|\xi|^2$ (in particular, $\mathrm{Re}s$ can be also negative, if $|\xi| > 0$).

Let us prove that (1.21) holds if $\widetilde{\boldsymbol{w}}$ satisfies the homogeneous jump conditions in (1.7). We have

$$(i\xi_1)^2\widetilde{w}_1 = i\xi_1(i\xi_1\widetilde{w}_1) = \frac{i\xi_1}{2}\widetilde{S}_{11}(\widetilde{\boldsymbol{w}}),$$
$$(i\xi_2)^2\widetilde{w}_1 = i\xi_2(i\xi_2\widetilde{w}_1 + i\xi_1\widetilde{w}_2) - i\xi_1(i\xi_2\widetilde{w}_2) = i\xi_2\widetilde{S}_{23}(\widetilde{w}) - \frac{i\xi_1}{2}\widetilde{S}_{22}(\widetilde{\boldsymbol{w}}),$$
$$\frac{d^2\widetilde{w}_1}{dy_3^2} = \frac{d}{dy_3}(\frac{d\widetilde{w}_1}{dy_3} + i\xi_1\widetilde{w}_3) - i\xi_1\frac{d\widetilde{w}_3}{dy_3} = \frac{d}{dy_3}\widetilde{S}_{13}(\widetilde{w}) - \frac{i\xi_1}{2}\widetilde{S}_{33}(\widetilde{w}),$$

which implies

$$\|\frac{d\widetilde{w}_1}{dy_3}\|^2 + |\xi|^2\|\widetilde{w}_1\|^2 = \int_{\mathbb{R}}(\sum_{j=1,2} i\xi_j\widetilde{w}_1 \cdot F_j(y_3) + \frac{d\widetilde{w}_1}{dy_3}F_3(y_3))dy_3,$$

where F_j are linear combinations of $\widetilde{S}_{km}(\widetilde{\boldsymbol{w}})$, in particular, $F_3 = \widetilde{S}_{13}(\widetilde{\boldsymbol{w}})$. Using the Cauchy inequality, we obtain

$$\|\frac{d\widetilde{w}_1}{dx_3}\| + |\xi|\|\widetilde{w}_1\| \le c\|\boldsymbol{F}\| \le c\|\widetilde{S}(\widetilde{w})\|.$$

The function $\widetilde{w}_2$ is estimated in a similar way, in addition, we have

$$|\xi_j|\|\widetilde{w}_3\| \le \|\widetilde{S}_{j3}(\widetilde{w})\| + \|\frac{d\widetilde{w}_j}{dy_3}\| \le c\|\widetilde{S}(\widetilde{\boldsymbol{w}})\|, \quad j = 1, 2,$$

and finally $\frac{d\widetilde{w}_3}{dy_3} = \frac{1}{2}\widetilde{S}_{33}(\widetilde{\boldsymbol{w}})$. This completes the proof of (1.21).

If $\mathrm{Re}s > 0$, then the above arguments are true for arbitrary non-zero $\xi \in \mathbb{R}^2$, in particular, the solution (1.18) of the problem (1.7) is unique in the class of functions $\widetilde{\boldsymbol{w}}^{\pm} \in W_2^2(\mathbb{R}^{\pm})$. It follows that D_0 is different from zero for arbitrary s with $\mathrm{Re}s > 0$ and $\xi \in \mathbb{R}^2$. In the opposite case the homogeneous system (1.14) and problem (1.7) would have non-zero solutions, which is not possible. Since D_0 is homogeneous, i.e., $D_0(\lambda\xi, \lambda^2 s) = \lambda^2 D_0(\xi, s)$, $\forall\lambda > 0$, there holds

$$|D_0(\xi, s)| \ge c(|s| + |\xi|^2) \tag{1.22}$$

with $c = \inf_{|\sigma|+|\eta|^2=1,\ \mathrm{Re}\sigma>0} |D_0(\sigma, \eta)| > 0$. Hence the same inequality (with another constant c) is satisfied, if $|\xi| \ge \pi/d_0$ and $\mathrm{Re}s$ is a negative number, small in comparison with π/d_0.

We pass to the analysis of a homogeneous problem (1.5). Elimination of $\widetilde{\vartheta}$ leads to

$$
\begin{cases}
\overline{\rho} s\widetilde{w}_\alpha - \Big(\sum\limits_{\beta=1,2} i\xi_\beta T_{\beta\alpha}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}T_{3\alpha}(\widetilde{\boldsymbol{w}})\Big) + \dfrac{\overline{\rho}p'(\overline{\rho})}{s} i\xi_\alpha\Big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{w}_\beta + \dfrac{d\widetilde{w}_3}{dy_3}\Big) = 0,\\
\overline{\rho} s\widetilde{w}_3 - \Big(\sum\limits_{\beta=1,2} i\xi_\beta T_{\beta 3}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dx_3}T_{33}(\widetilde{\boldsymbol{w}})\Big) + \dfrac{\overline{\rho}p'(\overline{\rho})}{s} \dfrac{d}{dy_3}\Big(\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{w}_\beta + \dfrac{d\widetilde{w}_3}{dy_3}\Big) = 0,\\
\pm y_3 > 0,\\
[\widetilde{\boldsymbol{w}}] = 0, \quad [\mu(\dfrac{d\widetilde{w}_\alpha}{dy_3} + i\xi_\alpha \widetilde{w}_3)] = \widetilde{b}_\alpha, \quad \alpha = 1,2,\\
[(2\mu + \mu_1 + \dfrac{\overline{\rho}p'(\overline{\rho}}{s})\dfrac{d\widetilde{w}_3}{dy_3} + (\mu_1 + \dfrac{\overline{\rho}p'(\overline{\rho})}{s})\sum\limits_{\beta=1,2} i\xi_\beta \widetilde{w}_\beta] = \widetilde{b}_3, \quad y_3 = 0,\\
\widetilde{\boldsymbol{w}}, \widetilde{\vartheta} \to 0, \quad |y_3| \to \infty,
\end{cases}
\tag{1.23}
$$

Clearly, (1.23) can be viewed as problem (1.7) with $\mu_1^\pm$ replaced by $\mu_1^\pm(s) = \mu_1^\pm + \frac{\overline{\rho}^\pm p'^\pm(\overline{\rho}^\pm)}{s}$ (and $\nu_1^\pm$ replaced by $\nu_1^\pm(s) = \nu_1^\pm + \frac{p'^\pm(\overline{\rho}^\pm)}{s}$). The expressions $1/(2\nu^\pm + \nu_1^\pm)$, $r_1^\pm(\xi, s) = \sqrt{s/(2\nu^\pm + \nu_1^\pm) + |\xi|^2}$, $e_1^\pm(y_3) = \frac{e^{\mp r_1^\pm y_3} - e^{\mp r^\pm y_3}}{r_1^\pm - r^\pm}$ go over into

$$
\frac{s}{a^\pm s + b^\pm}, \quad \sqrt{\frac{s^2}{a^\pm s + b^\pm} + |\xi|^2} \equiv r_{11}^\pm(\xi, s), \quad e_{11}^\pm(y_3) = \frac{e^{\mp r_{11}^\pm y_3} - e^{\mp r^\pm y_3}}{r_{11}^\pm - r^\pm}
$$

where

$$
a^\pm = 2\nu^\pm + \nu_1^\pm, \quad b^\pm = p^{\pm\prime}(\overline{\rho}^\pm) > 0, \quad r_{11}^\pm = \sqrt{\frac{s^2}{a^\pm s + b^\pm} + |\xi|^2}.
$$

We notice that all these expression are meaningful for negative $\mathrm{Re} s = s_1$, if $|\xi| > 0$.

We proceed with reducing (1.3) to a similar homogeneous problem (with $\boldsymbol{f} = 0, h = 0$).

We set $\boldsymbol{f}^\pm = 0$, $h^\pm = 0$ for $\pm y_3 > d_0$ and define $\boldsymbol{f}_*^\pm$ and $h_*^\pm$ as extensions of $\boldsymbol{f}^\pm$ and $h^\pm$, respectively, from $Q^\pm = \Omega' \times \mathbb{R}_\pm$ into $Q = Q^+ \cup Q^-$ with preservation of class. We set $\boldsymbol{u}^\pm = \boldsymbol{u}_1^\pm + \boldsymbol{u}_2^\pm$, $\sigma^\pm = \sigma_1^\pm + \sigma_2^\pm$, where $\boldsymbol{u}_1^\pm, \sigma_1^\pm$ satisfy

$$
\begin{cases}
\overline{\rho}\boldsymbol{u}_{1t}^\pm - \nabla\cdot T^\pm(\boldsymbol{u}_1^\pm) + p'(\overline{\rho})\nabla\sigma_1^\pm = \boldsymbol{f}_*^\pm,\\
\sigma_{1t}^\pm + \overline{\rho}\nabla\cdot \boldsymbol{u}_1^\pm = h_*^\pm, \quad y \in Q, \quad t > 0,\\
\boldsymbol{u}_1^\pm(y,0) = 0, \quad \sigma_1^\pm(y,0) = 0, \quad y \in Q,
\end{cases}
\tag{1.24}
$$

and $\boldsymbol{u}_2^+, \sigma_2^+$ solve the problem

$$
\begin{cases}
\overline{\rho}\boldsymbol{u}_{2t}^+ - \nabla\cdot T(\boldsymbol{u}_2^+) + p'(\overline{\rho}^+)\nabla\sigma_2^+ = 0,\\
\sigma_{2t}^+ + \overline{\rho}^+\nabla\cdot \boldsymbol{u}_2^+ = 0, \quad y \in Q^+, \quad t > 0,\\
\boldsymbol{u}_2^+ = -[\boldsymbol{u}_1], \quad y_3 = 0, \quad t > 0,\\
\boldsymbol{u}_2^+(y,0) = 0, \quad \sigma_2^+(y,0) = 0, \quad y \in Q^+,
\end{cases}
\tag{1.25}
$$

whereas $\boldsymbol{u}_2^- = 0$, $\sigma_2^- = 0$.

Making the Fourier transform with respect to all the space variables (y_1, y_2, y_3) and the Laplace transform with respect to t,

$$\mathcal{F}u = \widetilde{u} = \int_0^\infty e^{-st}dt \int_{\mathbb{R}} dy_3 \int_{\Omega'} u(y,t)e^{-y'\cdot\xi' - iy_3\xi_3}dy',$$

we reduce (1.24) to the algebraic system

$$\begin{cases} \overline{\rho}^{\pm}(s+\nu^{\pm}|\xi|^2)\widetilde{\boldsymbol{u}}_1^{\pm} + (\nu^{\pm}+\nu_1^{\pm})\boldsymbol{\xi}(\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{u}}_1^{\pm})) + p^{\pm'}(\overline{\rho}^{\pm})i\boldsymbol{\xi}\widetilde{\sigma}_1^{\pm} = \widetilde{\boldsymbol{f}}_*^{\pm}, \\ s\widetilde{\sigma}_1^{\pm} + \overline{\rho^{\pm}}(i\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{u}}_1^{\pm}) = \widetilde{h}_*^{\pm}, \quad |\xi|^2 = |\xi'|^3 + \xi_3^2. \end{cases} \tag{1.26}$$

We always assume that $\xi_\alpha = k_\alpha\pi/d_0$, $k_\alpha = \pm1, \pm2, ...,$ $\alpha = 1, 2$.

Elimination of $\widetilde{\sigma}_1^{\pm}$ leads to the system for $\widetilde{\boldsymbol{u}}_1^{\pm}$:

$$(s+\nu^{\pm}|\xi|^2)\widetilde{\boldsymbol{u}}_1^{\pm} + (\nu^{\pm}+\nu_1^{\pm}(s))\boldsymbol{\xi}(\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{u}}_1^{\pm}) = \frac{1}{\overline{\rho}^{\pm}}(\widetilde{\boldsymbol{f}}_*^{\pm} - \frac{p'(\overline{\rho}^{\pm})i\boldsymbol{\xi}\widetilde{h}_*^{\pm}}{s}) \equiv \widetilde{\boldsymbol{g}}^{\pm}, \tag{1.27}$$

the solution of which is given by

$$\widetilde{\boldsymbol{u}}_1^{\pm} = \mathcal{H}^{\pm}\widetilde{\boldsymbol{g}}^{\pm}/(s+\nu^{\pm}|\xi|^2), \tag{1.28}$$

where $\mathcal{H}^{\pm}$ is the matrix with the elements

$$H_{jk}^{\pm} = \delta_{jk} - \frac{(\nu^{\pm}+\nu_1^{\pm}(s))\xi_j\xi_k}{s+(2\nu^{\pm}+\nu_1^{\pm}(s))|\xi|^2} = \delta_{jk} - \frac{\xi_j\xi_k}{\frac{s}{a_1^{\pm}s+b^{\pm}}(s+\nu^{\pm}|\xi|^2)+|\xi|^2}. \tag{1.29}$$

We notice that

$$\mathcal{F}\nabla\cdot\boldsymbol{u}_1^{\pm} = i\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{u}}_1^{\pm} = \frac{i\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{g}}^{\pm}}{s+(2\nu^{\pm}+\nu_1^{\pm}(s))|\xi|^2} = \frac{s}{a_1^{\pm}s+b^{\pm}}\frac{i\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{g}}^{\pm}}{\frac{s}{a_1^{\pm}s+b^{\pm}}(s+\nu^{\pm}|\xi|^2)+|\xi|^2)}. \tag{1.30}$$

To estimate the solution of (1.2) near the rigid wall S, we need to consider one more model problem

$$\begin{cases} \overline{\rho}\boldsymbol{w}_t - \nabla\cdot T(\boldsymbol{w}) + p'(\overline{\rho})\nabla\vartheta = \boldsymbol{f}, \\ \vartheta_t + \overline{\rho}\nabla\cdot\boldsymbol{w} = h, \quad y\in Q^+, \quad t>0, \\ \boldsymbol{w}(y',t) = \boldsymbol{a}(y',t), \quad y'\in\Omega', \quad t>0, \\ \boldsymbol{w}(y,0) = 0, \quad \vartheta(y,0) = 0, \quad y\in Q^+ \end{cases} \tag{1.31}$$

with $\overline{\rho} = \overline{\rho}^-$, $T(\boldsymbol{w}) = T^-(\boldsymbol{w})$, under the assumption supp$\boldsymbol{w}$, supp$\vartheta \subset Q^+$.

Let $\boldsymbol{f} = 0$, $h = 0$. The Fourier-Laplace transform (1.4) with respect to y_1, y_2, t converts (1.31) into

$$\begin{cases} \overline{\rho}s\widetilde{w}_\alpha - (\sum\limits_{\beta=1,2} i\xi_\beta\widetilde{T}_{\beta\alpha}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}\widetilde{T}_{3\alpha}(\widetilde{\boldsymbol{w}})) + p'(\overline{\rho})i\xi_\alpha\widetilde{\vartheta} = 0, \quad \alpha = 1,2, \\ \overline{\rho}s\widetilde{w}_3 - (\sum\limits_{\beta=1,2} i\xi_\beta\widetilde{T}_{\beta3}(\widetilde{\boldsymbol{w}}) + \dfrac{d}{dy_3}\widetilde{T}_{33}(\widetilde{\boldsymbol{w}})) + p'(\overline{\rho})\dfrac{d}{dy_3}\widetilde{\vartheta} = 0, \\ s\widetilde{\vartheta} + \overline{\rho}(\sum\limits_{\beta=1,2}(i\xi_\beta\widetilde{w}_\beta + \dfrac{d}{dy_3}\widetilde{w}_3) = 0, \quad y_3>0, \\ \widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{a}}, \quad y_3 = 0, \quad \widetilde{\boldsymbol{w}}\to 0, \quad y_3\to+\infty. \end{cases} \tag{1.32}$$

We also consider the problem

$$\begin{cases} \bar\rho s\widetilde w_\alpha - (\sum_{\beta=1,2} i\xi_\beta \widetilde T_{\beta\alpha}(\widetilde{\boldsymbol w}) + \dfrac{d}{dy_3}\widetilde T_{3\alpha}(\widetilde{\boldsymbol w})) = 0, \quad \alpha = 1,2, \\ \bar\rho s\widetilde w_3 - (\sum_{\beta=1,2} i\xi_\beta \widetilde T_{\beta 3}(\widetilde{\boldsymbol w}) + \dfrac{d}{dy_3}\widetilde T_{33}(\widetilde{\boldsymbol w})) = 0, \quad y_3 > 0, \\ \widetilde{\boldsymbol w} = \widetilde{\boldsymbol a}, \quad y_3 = 0, \quad \widetilde{\boldsymbol w} \to 0, \quad y_3 \to +\infty. \end{cases} \tag{1.33}$$

We assume that $\xi = (\pi k_1/d_0, \pi k_2/d_0)$, $k = \pm1, \pm2, \ldots$. The solution of (1.33) is given by the formula similar to (1.9), i.e.,

$$\widetilde{\boldsymbol w}(\xi, s, y_3) = \begin{pmatrix} h_1 r + i\xi_1 h_3 \\ h_2 r + i\xi_2 h_3 \\ H - r_1 h_3 \end{pmatrix} e^{-ry_3} + h_3(r_1 - r)\begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1 \end{pmatrix} \frac{e^{-r_1 y_3} - e^{-ry_3}}{r_1 - r}, \quad y_3 > 0, \tag{1.34}$$

where $r = \sqrt{s/\nu^- + |\xi|^2}$, $r_1 = \sqrt{s/(2\nu^- + \nu_1^-) + |\xi|^2}$, $H = \sum_{\beta=1,2} i\xi_\beta h_\beta$. (see [2]). The boundary conditions imply

$$h_\alpha r + i\xi_\alpha h_3 = \widetilde a_\alpha, \quad \alpha = 1,2,$$
$$Hr - \xi^2 h_3 = \widetilde A = \sum_{\beta=1}^{2} i\xi_\beta \widetilde a_\beta,$$
$$H - h_3 r_1 = \widetilde a_3,$$

hence

$$\begin{aligned} & h_3 = \frac{\widetilde A - r\widetilde a_3}{rr_1 - |\xi|^2}, \quad H = \widetilde a_3 + r_1 \frac{\widetilde A - r\widetilde a_3}{rr_1 - |\xi|^2}, \\ & h_3(r_1 - r) = \frac{(\widetilde A - r\widetilde a_3)(r_1 - r)}{rr_1 - |\xi|^2} = \frac{\widetilde A - r\widetilde a_3}{r + r_1}\frac{s(\frac{1}{2\nu+\nu_1} - \frac{1}{\nu})\nu}{sR^-} = -\frac{\nu + \nu_1}{2\nu + \nu_1}\frac{\widetilde A - r\widetilde a_3}{r_1 + \varkappa^- r}, \end{aligned} \tag{1.35}$$

so (1.34) is equivalent to

$$\widetilde{\boldsymbol w}(\xi, s.y_3) = \begin{pmatrix} \widetilde a_1 \\ \widetilde a_2 \\ \widetilde a_3 \end{pmatrix} e^{-ry_3} + h_3(r_1 - r)\begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_1 \end{pmatrix} \frac{e^{-r_1 y_3} - e^{-ry_3}}{r_1 - r} \quad y_3 > 0, \tag{1.36}$$

with $h_3(r_1 - r)$ given in (1.35).

The solution of (1.32) is obtained by replacing ν_1 with $\nu_1(s)$ in (1.36), which yields

$$\begin{aligned} & \widetilde{\boldsymbol w}(\xi, s.y_3) = \begin{pmatrix} \widetilde a_1 \\ \widetilde a_2 \\ \widetilde a_3 \end{pmatrix} e^{-ry_3} + h_3(r_{11} - r)\begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -r_{11} \end{pmatrix} \frac{e^{-r_{11} y_3} - e^{-ry_3}}{r_{11} - r} \quad y_3 > 0, \\ & h_3(r_{11} - r) = -\frac{a_1 s + b}{as + b}\frac{\widetilde A - r\widetilde a_3}{r_{11} + \varkappa_1(s) r}, \quad \varkappa_1(s) = \frac{\nu s}{as + b}, \quad r_{11} = \sqrt{\frac{s^2}{as + b} + |\xi|^2}, \\ & a_1 = \nu^- + \nu_1^-, \quad a = 2\nu^- + \nu_1^-, \quad b = \bar\rho^- p'(\bar\rho^-). \end{aligned} \tag{1.37}$$

To reduce (1.31) to a similar homogeneous problem, we introduce auxiliary functions $\boldsymbol{u}_1$ and σ_2 satisfying the relations analogous to (1.24):

$$\begin{cases} \overline{\rho}\boldsymbol{u}_{1t} - \nabla \cdot T(\boldsymbol{u}_1^{\pm}) + p'(\overline{\rho})\nabla\sigma_1 = \boldsymbol{f}^*, \\ \sigma_{1t} + \overline{\rho}\nabla \cdot \boldsymbol{u}_1 = h^*, \\ \boldsymbol{u}_1(y,0) = 0, \quad \sigma_1(y,0) = 0, \quad y \in Q. \end{cases} \tag{1.38}$$

with $\overline{\rho} = \overline{\rho}^-$ and with $\boldsymbol{f}^*$ and h^* defined as the extensions of $\boldsymbol{f}$ and h into $Q = \Omega' \times \mathbb{R}$ with preservation of class. By the Fourier-Laplace transform with respect to y_1, y_2, y_3, t (we assume that $|\xi'| \geq \pi/d_0$) equations (1.38) reduce to

$$\begin{cases} \overline{\rho}(s + \nu|\xi|^2)\widetilde{\boldsymbol{u}}_1 + (\nu + \nu_1)\boldsymbol{\xi}(\boldsymbol{\xi} \cdot \widetilde{\boldsymbol{u}}_1) + p'(\overline{\rho})i\boldsymbol{\xi}\widetilde{\sigma_1} = \widetilde{\boldsymbol{f}^*}, \\ s\widetilde{\sigma}_1 + \overline{\rho}(i\boldsymbol{\xi} \cdot \widetilde{\boldsymbol{u}}_1) = \widetilde{h}^*, \quad |\xi|^2 = |\xi'|^2 + \xi_3^2, \end{cases} \tag{1.39}$$

which implies

$$(s + \nu|\xi|^2)\widetilde{\boldsymbol{u}}_1 + (\nu + \nu_1(s))\boldsymbol{\xi}(\boldsymbol{\xi} \cdot \widetilde{\boldsymbol{u}}_1) = \frac{1}{\overline{\rho}}(\widetilde{\boldsymbol{f}}^* - \frac{p'(\overline{\rho})i\boldsymbol{\xi}\widetilde{h}^*}{s}) \equiv \widetilde{\boldsymbol{g}}. \tag{1.40}$$

As above, we have

$$\widetilde{\boldsymbol{u}}_1 = \mathcal{H}\widetilde{\boldsymbol{g}}/(s + \nu|\xi|^2), \tag{1.41}$$

where $\mathcal{H} = (H_{jk})_{j,k=1,2,3}$,

$$H_{jk} = \delta_{jk} - \frac{\xi_j\xi_k}{\frac{s}{a_1 s + b}(s + \nu|\xi|^2) + |\xi|^2}. \tag{1.42}$$

Problems (1.5), (1.32) with $\xi = 0$ are treated in Section 2.

2 L_p-estimates.

In this section, we prove L_p-estimates for solutions of model problems studied in Section 1.

We obtain the following result.

Theorem 1. *The functions $\boldsymbol{w}$ and ϑ with supp$\boldsymbol{w}$, supp$\vartheta \subset \overline{\Omega}$ satisfying (1.3) with $\boldsymbol{b}(y',t)$ and vanishing for $t = 0$ are subject to the inequality*

$$\begin{aligned} &\sum_{\pm}(\|e^{\beta t}\boldsymbol{w}\|_{W_p^{2,1}(Q_\infty^\pm)} + \|e^{\beta t}\vartheta\|_{W_p^{1,0}(Q_\infty^\pm)} + \|e^{\beta t}\vartheta_t\|_{W_p^{1,0}(Q_\infty^\pm)}) \\ &\leq c\Big(\sum_{\pm}(\|e^{\beta t}\boldsymbol{f}\|_{L_p(Q_\infty^\pm)} + \|e^{\beta t}h\|_{W_p^{1,0}(Q_\infty^\pm)}) + \|e^{\beta t}\boldsymbol{b}\|_{W_p^{1-1/p,1/2-1/2p}(\Omega'_\infty)}\Big), \end{aligned} \tag{2.1}$$

where $Q_\infty^\pm = \Omega^\pm \times (0,\infty)$, $Q'_\infty = \Omega' \times (0,\infty)$, $\beta > 0$.

By $W_p^{2,1}(\Omega \times (0,T))$, $\Omega \subset \mathbb{R}^n$, we mean the space $W_p^{2,0}(\Omega \times (0,T)) \cap W_p^{0,1}(\Omega \times (0,T))$, where

$$W_p^{2,0}(\Omega \times (0,T)) = L_p(0,T; W_p^2(\Omega)), \quad W_p^{0,1}(\Omega \times (0,T)) = W_p^1(0,T; L_p(\Omega)).$$

The norm in isotropic space $W_p^l(\Omega)$ is given by

$$\|u\|_{W_p^l(\Omega)}^p = \sum_{|j|\leq l}\int_\Omega |D^j u(x)|^p dx,$$

if l is an integer, and

$$\|u\|^p_{W^l_p(\Omega)} = \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^p}{|x-y|^{n+p\lambda}} dx,$$

if $l = [l] + \lambda$, $0 < \lambda < 1$. On the manifolds, the above spaces are defined in a standard way, i.e., with the help of the partition of unity and local maps.

The proof of Theorem 1 occupies the major part of the section. We start with estimates of $\boldsymbol{u}_1$ and σ_1 satisfying (1.26), (1.27). We assume that

$$\|e^{\beta t} \boldsymbol{f}^\pm_* \|_{L_p(Q_\infty)} \le c \|e^{\beta t} \boldsymbol{f}^\pm\|_{L_p(Q^\pm_\infty)}, \quad \|e^{\beta t} h^\pm_* \|_{W^{1,0}_p(Q_\infty)} \le c \|e^{\beta t} h^\pm \|_{W^{1,0}_p(Q^\pm_\infty)}, \tag{2.2}$$

where $Q_\infty = Q^+_\infty \cup Q^-_\infty$. The estimate of $\boldsymbol{u}_1$ is carried out on the basis of the Marcinkiewicz-Mikhlin-Lizorkin theorem:

Theorem 2 [5,6,7]. *Assume that* $\widetilde{u} = \mathcal{F} u$, $\widetilde{v} = \mathcal{F} v$ *and*

$$\widetilde{v} = m(\xi, s) \widetilde{u}$$

where $\mathcal{F}$ *is the Fourier-Laplace transformation* $\mathcal{F} u(\xi, s) = \int_0^\infty e^{-st} dt \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x,t) dx$ *If*

$$M_p(m) \equiv \sup_{\mathrm{Re} s = s_1} \sup_{\xi \in \mathbb{R}^n} |\mathcal{M} m| \le c \tag{2.3}$$

where

$$\begin{aligned} \mathcal{M} m = \Big(|m(\xi,s)| &+ \sum_{j=1}^n \sum_{k_i \ne k_j} |\xi_{k_1} \ldots \xi_{k_j}| \Big|\frac{\partial^j m}{\partial \xi_{k_1} \ldots \partial \xi_{k_j}}\Big| \\ &+ |s| \Big|\frac{\partial m}{\partial s}\Big| + \sum_{j=1}^n \sum_{k_i \ne k_j} |s| |\xi_{k_1} \ldots \xi_{k_j}| \Big| \frac{\partial^{j+1} m}{\partial s \partial \xi_{k_1} \ldots \partial \xi_{k_j}} \Big|, \end{aligned} \tag{2.4}$$

then

$$\|e^{\beta t} v\|_{L_p(\mathbb{R}^n \times \mathbb{R}_+)} \le c M_p(m) \|e^{\beta t} u\|_{L_p(\mathbb{R}^n \times \mathbb{R}_+)}, \quad \beta = -\mathrm{Re} s.$$

The function m with these properties is referred to as L_p Fourier multiplier; the set of the L_p-multipliers is denoted by $\mathfrak{M}$.

The Marcinkiewicz-Mikhlin-Lizorkin theorem applies also to the Fourier series, if the Fourier coefficients are defined for all real values of ξ (see [8]). By repeating the arguments in [7] it is easy to show that the condition (2.3) should be replaced in our case with

$$\begin{aligned} M_p(m) = \sup_{\mathrm{Re} s = s_1} \sup_{|\xi| \ge \pi/d_0} \Big(|m(\xi,s)| &+ \sum_{j=1}^3 |\xi_j \frac{\partial m}{\partial \xi_j}| + \sum_{j \ne k} |\xi_j \xi_k \frac{\partial^2 m(\xi,s)}{\partial \xi_1 \partial \xi_2}| \\ + |\xi_1 \xi_2 \xi_3 \frac{\partial^3 m}{\partial \xi_1 \partial \xi_2 \partial \xi_3}| &+ \sum_{j=1}^3 |s| |\xi_j \frac{\partial^2 m(\xi,s)}{\partial s \partial \xi_j}| \\ + \sum_{j \ne k} |s| |\xi_j| |\xi_k| |\frac{\partial^3 m(\xi,s)}{\partial s \partial \xi_j \partial \xi_k}| &+ |s| |\xi_1 \xi_2 \xi_3 \frac{\partial^4 m}{\partial s \partial \xi_1 \partial \xi_2 \partial \xi_3}| \Big) < \infty, \quad |\xi|^2 = |\xi'|^2 + \xi_3^2. \end{aligned} \tag{2.5}$$

for $\widetilde{m} = \widetilde{H}_{jk}$.

Let us consider the expression

$$\mathbb{P}^{\pm}(\xi,s)=\frac{s}{a^{\pm}s+b^{\pm}}(s+\nu^{\pm}|\xi|^2)+|\xi|^2=\frac{sa^{\pm}|s|^2+b^{\pm}s^2}{|a^{\pm}s+b^{\pm}|^2}+\frac{s\nu^{\pm}}{a^{\pm}s+b^{\pm}}|\xi|^2+|\xi|^2.$$

Since $|\frac{s_1a^{\pm}|s|^2+b^{\pm}s^2}{a^{\pm}s+b|^2}|$ is bounded by a constant independent of s_2, there holds $|\mathbb{P}|\geq c(|s+|\xi|^2)$, if d_0 is small. Moreover,

$$|\mathcal{M}\mathbb{P}^{-1}|\leq c|\mathbb{P}^{-1}|,\quad M_p(\xi_j\xi_k\mathbb{P}^{-1})+M_p(s\mathbb{P}^{-1})\leq c.$$

We represent $(\widetilde{\boldsymbol{u}}_1^{\pm},\widetilde{\sigma}_1)$ as the sum

$$\widetilde{\boldsymbol{u}}_1^{\pm}=\widetilde{\boldsymbol{w}}_1+\widetilde{\boldsymbol{w}}_2,\quad \widetilde{\sigma}_1^{\pm}=\widetilde{\phi}_1+\widetilde{\phi}_2,$$

where $\widetilde{\boldsymbol{w}}_i,\widetilde{\phi}_i$ satisfy (1.26) with $\widetilde{h}_*^{\pm}=0$ and $\widetilde{\boldsymbol{f}}_*^{\pm}=0$, respectively. Hence $\widetilde{\boldsymbol{w}_1}$ is given by the formula (1.28) with $\frac{1}{\bar{\rho}^{\pm}}\widetilde{\boldsymbol{f}}_*^{\pm}$ instead of $\widetilde{\boldsymbol{g}}$. Applying Theorem 2, we obtain

$$\|e^{\beta t}\boldsymbol{w}_1\|_{W_p^{2,1}(Q_\infty)}\leq c\|e^{\beta t}F^{-1}\mathcal{H}\widetilde{\boldsymbol{f}}_*^{\pm}\|_{L_p(Q_\infty)}\leq c\|e^{\beta t}\boldsymbol{f}^{\pm}\|_{L_p(Q_\infty^{\pm})},\tag{2.6}$$

moreover, from the system of equations (1.26) for $\widetilde{\boldsymbol{w}}_1$ and $\widetilde{\phi}_1$, we deduce

$$\begin{aligned}&\|e^{\beta t}\phi_{1t}\|_{W_p^{1,0}(Q_\infty)}+\|e^{\beta t}\nabla\phi_1\|_{W_p^{1,0}(Q_\infty)}\\&\leq c(\|e^{\beta t}\boldsymbol{w}_1\|_{W_p^{2,1}(Q_\infty)}+\|e^{\beta t}\boldsymbol{f}_*^{\pm}\|_{L_p(Q_\infty)})\leq c\|e^{\beta t}\boldsymbol{f}^{\pm}\|_{L_p(Q_\infty^{\pm})}.\end{aligned}$$

Since $\int_{\Omega'}\widetilde{\phi}_1dy'=0$, the function $\widetilde{\phi}_1$ satisfies the same estimate.
We pass to estimates of $(\boldsymbol{w}_2,\phi_2)$. By (1.30),

$$i\boldsymbol{\xi}\cdot\widetilde{\boldsymbol{w}}_2=\frac{p^{\pm'}(\bar{\rho}^{\pm})}{\bar{\rho}^{\pm}(a_1^{\pm}s+b^{\pm})}\frac{|\xi|^2\widetilde{h}_*^{\pm}}{\frac{s}{a_1^{\pm}s+b^{\pm}}(s+\nu^{\pm}|\xi|^2)+|\xi|^2)}\equiv\widetilde{\chi},$$

and, by Theorem 2,

$$\|e^{\beta t}\chi\|_{W_p^{1,0}(Q_\infty)}+\|e^{\beta t}\chi_t\|_{W_p^{1,0}(Q_\infty)}\leq c\|e^{\beta t}h_*^{\pm}\|_{W_p^{1,0}(Q_\infty)}\leq c\|e^{\beta t}h^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}.$$

The functions ($\widetilde{\boldsymbol{w}}_2$ and $\widetilde{\phi}_2$) can be viewed as a solution of the transformed Stokes system:

$$\begin{cases}\bar{\rho}^{\pm}(s+\nu^{\pm}|\xi|^2)\widetilde{\boldsymbol{w}}_2^{\pm}+p^{\pm'}(\bar{\rho}|^{\pm})i\boldsymbol{\xi}\widetilde{\phi}_2=-(\nu^{\pm}+\nu_1^{\pm})i\boldsymbol{\xi}\widetilde{\chi},\\ i\widetilde{\boldsymbol{\xi}}\cdot\boldsymbol{w}_2^{\pm}=\widetilde{\chi}.\end{cases}$$

It can be easily solved; the solution is given by

$$\widetilde{\boldsymbol{w}}_2^{\pm}=-\frac{i\boldsymbol{\xi}}{|\xi|^2}\widetilde{\chi},\quad \widetilde{\phi}_2=-\frac{1}{p^{\pm'}(\bar{\rho}^{\pm})}((\nu^{\pm}+\nu_1\pm)\widetilde{\chi}+\frac{\bar{\rho}^{\pm}(s+\nu^{\pm}|\xi|^2|)}{|\xi|^2}\widetilde{\chi}).$$

From these formulas and from $s\widetilde{\phi}_2=\widetilde{h}-\bar{\rho}^{\pm}\widetilde{\chi}$ we deduce

$$\begin{aligned}&\|e^{\beta t}\boldsymbol{w}_2\|_{W_p^{2,1}(Q_\infty)}+\|e^{\beta t}\nabla\phi_2\|_{W_p^{1,0}(Q_\infty)}++\|e^{\beta t}\nabla\phi_{2t}\|_{W_p^{1,0}(Q_\infty)}\\&\leq c(\|e^{\beta t}\chi\|_{W_p^{1,0}(Q_\infty)}+\|e^{\beta t}\chi_t\|_{W_p^{1,0}(Q_\infty)}+\|e^{\beta t}h^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}))\leq c\|e^{\beta t}h^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}.\end{aligned}$$

Collecting estimates, we arrive at

$$\begin{aligned}&\sum_{\pm}(\|e^{\beta t}\boldsymbol{u}_1^{\pm}\|_{W_p^{2,1}(Q_\infty^{\pm})}+\|e^{\beta t}\sigma_1^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}+\|e^{\beta t}\sigma_{1t}^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})})\\&\le c\sum_{\pm}(\|\boldsymbol{f}^{\pm}\|_{L_p(Q_\infty^{\pm})}++\|h^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}).\end{aligned}\tag{2.7}$$

The problem (1.25) is treated below; we show that

$$\begin{aligned}&\|e^{\beta t}\boldsymbol{u}_2\|_{W_p^{2,1}(Q_\infty^{\pm})}+\|e^{\beta t}\sigma_2^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}+\|e^{\beta t}\sigma_{2t}^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})})\\&\le c\|e^{\beta t}[\boldsymbol{u}_1]\|_{W_p^{2-1/p,1-1/2p}(Q'_\infty)}\le c\sum_{\pm}\|e^{\beta t}\boldsymbol{u}_1^{\pm}\|_{W_p^{2,1}(Q_\infty^{\pm})}.\end{aligned}\tag{2.8}$$

Hence

$$\begin{aligned}&\sum_{\pm}(\|e^{\beta t}\boldsymbol{u}^{\pm}\|_{W_p^{2,1}(Q_\infty^{\pm})}+\|e^{\beta t}\sigma^{\pm}\|_{W_p^{2,1}(Q_\infty^{\pm})}+\|e^{\beta t}\sigma_t^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})})\\&\le c\sum_{\pm}(\|e^{\beta t}\boldsymbol{f}_1\|_{L_p(Q_\infty^{\pm})}+\|e^{\beta t}h^{\pm}\|_{W_p^{1,0}(Q_\infty^{\pm})}).\end{aligned}\tag{2.9}$$

The differences $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{u}$, $\vartheta=\theta-\sigma$ solve the homogeneous problem (1.3) (with $\boldsymbol{f}=0, h=0$).

We go back to (1.7). The solution of this problem is expressed by (1.18). We again make use of the Marcinkiewicz-Mikhlin-Lizorkin theorem for functions given in $\{y'\in\mathfrak{Q}',\ t>0\}$, vanishing for $t<0$ and periodic with respect to y'. In this case the condition (2.3) should be taken in the form

$$M_p(m)\equiv\sup_{\mathrm{Re}s=s_1}\sup_{|\xi|\ge\pi/d_0}|\mathcal{M}m|\le c,\tag{2.10}$$

$$\begin{aligned}\mathcal{M}m&=|m(\xi,s)|+\sum_{j=1}^{n}\sum_{k_i\neq k_j}|\xi_{k_1}...\xi_{k_j}\frac{\partial^j m}{\partial\xi_{k_1}...\partial\xi_{k_j}}|\\&+|s\frac{\partial m}{\partial s}|+\sum_{j=1}^{n}\sum_{k_i\neq k_j}|s\xi_{k_1}...\xi_{k_j}\frac{\partial^{j+1}m}{\partial s\partial\xi_{k_1}...\partial\xi_{k_j}}|,\quad n=2.\end{aligned}\tag{2.11}$$

Moreover, we make use of the following proposition.

Proposition 1. *Assume that $P_0(\xi,s)/r^+\in\mathfrak{M}$, $P_1(\xi,s)/r^{+2}\in\mathfrak{M}$ (this means that $m=P_0/r^+, P_1r^{+2}$ are subject to (2.10)). Then the functions $w_0^+(x,t)=F^{-1}P_0e^{-r^+y_3}Fd_0^+$ and $w_1^+(x,t)=F^{-1}P_1e_1^+(y_3)Fd_1^+$, where $e_1^+=\frac{e^{-r_1^+y_3}-e^{-r^+y_3}}{r_1^+-r^+}$ and $d_i(x',0)=0$ (in the case $p>3$), satisfy*

$$\|e^{\beta t}w_i^+\|_{L_p(Q_\infty^+)}\le c\|e^{\beta t}d_i\|_{W_p^{1-1/p,1/2-1/(2p)}(Q'_\infty)},\quad i=0,1.\tag{2.12}$$

Proof. Following Volevich [9], we represent w_i as follows:

$$\begin{aligned}\widetilde{w}_0^+&=-\int_0^\infty P_0(\xi,s)\frac{d}{dy_3}(e^{-r^+(x_3+y_3)}\widetilde{d}_0^+(\xi,s,y_3))dy_3=2\int_0^\infty P_0e^{-r^+(x_3+y_3)}r^+\widetilde{d}_0^+(\xi,s,y_3)dy_3,\\\widetilde{w}_1^+&=2\int_0^\infty e_1^+(x_3+y_3)P_1r^+\widetilde{d}_1^+(\xi,s,y_3)dy_3+\int_0^\infty P_1e^{-r^+(x_3+y_3)}\widetilde{d}_1^+(\xi,s,y_3)dy_3,\end{aligned}$$

where

$$\widetilde{d}_i^+(\xi,s,y_3)=\widetilde{d}_i^+(\xi,s)e^{-ry_3}.$$

Since Re$s=s_1<0$ and $d_i^+|_{t=0}=0$, there holds

$$\|e^{\beta t}F^{-1}r^+\widetilde{d}_i^+\|_{L_p(Q_\infty^\pm)}\le c\|e^{\beta t}d_i^+\|_{W_p^{1-1/p,1/2-1/2}(Q'_\infty)}.$$

Now we use the inequalities

$$|M_pr^+e^{-r^+t}|+|M_pr^{+2}e_1^+(t)|\le\frac{c}{t},\quad t>0,$$

obtained in [10] (or by direct calculation) and get

$$\|e^{\beta t}w_0^+\|_{L_p(Q'_\infty)}\le c\int_0^\infty\|e^{\beta t}F^{-1}P_0\widetilde{d}_0^+\|_{L_p(Q'_\infty)}\frac{dy_3}{x_3+y_3},$$
$$\|e^{\beta t}w_1^+\|_{L_p(Q'_\infty)}\le c\int_0^\infty\|e^{\beta t}F^{-1}P_1/r^+\widetilde{d}_1^+\|_{L_p(Q'_\infty)}\frac{dy_3}{x_3+y_3}.$$

From our assumptions concerning P_0 and P_1 and the continuity of the Hilbert transform, it follows

$$\begin{aligned}&\|e^{\beta t}F^{-1}P_0\widetilde{d}_0\|_{L_p(Q_\infty^\pm)}\le c\|e^{\beta t}F^{-1}r^+\widetilde{d}_0^+\|_{L_p(Q_\infty^\pm)}\le c\|e^{\beta t}d_0^+\|_{W_p^{1-1/p,1/2-1/2}(Q'_\infty)},\\&\|e^{\beta t}F^{-1}P_1\widetilde{d}_1^+\|_{L_p(Q_\infty^\pm)}\le c\|e^{\beta t}F^{-1}r^+\widetilde{d}_1^+\|_{L_p(Q_\infty^\pm)}\le c\|e^{\beta t}d_1^+\|_{W_p^{1-1/p,1/2-1/2}(Q'_\infty)},\end{aligned}\tag{2.13}$$

q.e.d. The functions $w_0^-(x,t)=F^{-1}P_0e^{r^-y_3}Fd_0^-$ and $w_1^-(x,t)=F^{-1}P_1e_1^-(y_3)Fd_1^-$ satisfy similar inequalities.

Using Proposition 1, formula (1.18), inequality (1.22), the relations

$$\frac{de_1^\pm}{dy_3}=-r_1^\pm e^{-r_1^\pm}-r^\pm e_1^\pm(y_3)$$

and

$$\begin{aligned}&|\mathcal{M}r|+|\mathcal{M}r_1|\le c|r|,\quad|\mathcal{M}r^{-1}|+|\mathcal{M}r_1^{-1}|\le c|r|^{-1},\\&M_p(R^\pm)\le c,\quad|\mathcal{M}D_0^{-1}|\le c|r|^{-2},\end{aligned}\tag{2.14}$$

we obtain the desired estimate for the solution of a homogeneous problem (1.3):

$$\sum_\pm\|e^{\beta t}\boldsymbol{w}^\pm\|_{W_p^{2,1}(Q_\infty^\pm)}\le c\|e^{\beta t}\boldsymbol{b}\|_{W_p^{1-1/p,1/2-1/2p}(Q'_\infty)}.\tag{2.15}$$

Now, we turn to the problem (1.3), (1.7), (1.23). The solution of (1.23) is also given by (1.18), but with $\nu_1^\pm(s)$ and $r_{11}^\pm$ instead of $\nu_1^\pm$ and $r_1^\pm$, respectively. We show that the assumptions of Proposition 1 still hold.

We consider $r_{11}=\sqrt{\frac{s^2}{as+b}+|\xi|^2}$ and $R_1=\frac{r_{11}+\varkappa_1(s)r}{r_{11}+r}$ (omitting indices $\pm$), where $\varkappa_1(s)=\frac{\nu s}{as+b}$. We have

$$r_{11}^2=\frac{s_1a|s|^2+b(s_1^2-s_2^2)}{|as+b|^2}+is_2\frac{a|s|^2+2bs_1}{|as+b|^2}+|\xi|^2.$$

Since $\frac{s_1 a|s|^2+b(s_1^2-s_2^2)}{|as+b|^2}$ and $\frac{a|s|^2+2bs_1}{|as+b|^2}$ are uniformly bounded by constants independent of s_2, there hold

$$c_1(|s|+|\xi|^2) \le |r_{11}^2| \le c_2(|s|+|\xi|^2)$$

and

$$c_3(|s|+|\xi|^2)^{1/2} \le \mathrm{Re} r_{11} \le |r_{11}| \le c_4(|s|+|\xi|^2)^{1/2}, \tag{2.16}$$

if d_0 is small. Moreover, easy calculations show that

$$\begin{aligned} &|\mathcal{M} r_{11}| \le c|r| \le c(|s|+|\xi|^2)^{1/2}, \\ &|r_1 - r_{11}| = \frac{|r_1^2 - r_{11}^2|}{|r_1 + r_{11}|} \le c\frac{cs}{(as+b)(r_1+r_{11})} \le \frac{c|r|}{|as+b|} \le \delta |r|, \\ &|\varkappa_1(s) - \varkappa| \le c\delta, \quad \delta \ll 1 \end{aligned} \tag{2.17}$$

if $|s| \ge h$ ($\delta \to 0$, if h grows without limits; we choose h in an appropriate way). This implies

$$|R_1 - R| \le c\delta, \quad c_5 \le |R_1| \le c_6. \tag{2.18}$$

The function D_0 goes over into

$$\begin{aligned} D_1 = &(-\mu^+(r^{2+}+|\xi|^2) + \mu^-(r^{2-}+|\xi|^2))(-\frac{R_1^-}{\mu^-}(1-2R_1^+) + \frac{R_1^+}{\mu^+}(1-2R_1^-)) \\ &+ 2(\mu^+ r^+ + \mu^- r^-)(\frac{r_{11}^+ R_1^-}{\mu^-} + \frac{r_{11}^- R_1^+}{\mu^+}) + (r^+ + r^-)(r_{11}^+(1-2R_1^-) + r_{11}^-(1-2R_1^+)). \end{aligned} \tag{2.19}$$

Making use of (2.16) - (2.19) we obtain

$$|D_1 - D_0| \le \delta |r|^2, \quad c'|r|^2 \le |D_1| \le c''|r^2|$$

and

$$|\mathcal{M} D_1| \le c|D_1|, \quad |\mathcal{M} D^{-1}| \le c|D_1|^{-1}, \tag{2.20}$$

provided $|s| \ge h$. If $|s| \le h$ and d_0 is small, then $|s| \le \delta_1 |\xi|^2$ with a small δ_1. In this case (2.20) can be proved by comparing $D_1(\xi, s)$ with $D_1(\xi, 0)$. It is easily seen that $R_1^{\pm}(\xi, 0) = 1/2$ and $D_1(\xi, 0) = |\xi|^2(\mu^+ + \mu^-)(\frac{1}{\mu^+} + \frac{1}{\mu^-})$, hence $|D_1(\xi, s) - D_1(\xi, 0)| \le c\delta_1 |\xi|^2$, $|D_1(\xi, s)| \ge c|r|^2$, if δ_1 is small. This shows that (2.20) hold also for $|s| \le h$.

The above arguments prove that the assumptions of proposition 1 are satisfied also for $D_y^j D_t^k \boldsymbol{w}(y,t)$, where $\widetilde{\boldsymbol{w}}$ is a solution of (1.23). Hence $\boldsymbol{w}$ satisfies (2.15). The estimate of θ is deduced from the equations

$$\begin{cases} \bar\rho \boldsymbol{v}_t - \nabla \cdot T(\boldsymbol{v}) + p'(\bar\rho)\nabla\theta = 0, \\ \theta_t + \bar\rho \nabla \cdot \boldsymbol{v} = 0, \quad y \in \mathbb{R}_+^3 \cup \mathbb{R}_-^3 \end{cases}$$

and as a result we obtain

$$\sum_{\pm}(\|e^{\beta t}\boldsymbol{w}\|_{W_p^{2,1}(Q_\infty^\pm)}) + \|e^{\beta t}\theta\|_{W_p^{1,0}(Q_\infty^\pm)} + \|e^{\beta t}\theta_t\|_{W_p^{1,0}(Q_\infty^\pm)}) \le c\|e^{\beta t}\boldsymbol{b}\|_{W_p^{1-1/p,1/2-1/2p}(Q'_\infty)}, \tag{2.21}$$

where $\mathfrak{Q}_\infty^\pm = \mathfrak{Q}^\pm \times (0,\infty)$, $\mathfrak{Q}_\infty^{\pm'} = \mathfrak{Q}^{\pm'} \times (0,\infty)$.

To complete the estimate of the solution of (1.3), we need to analyze the case $\xi = 0$ in (1.7), (1.26), i.e., to estimate the zero mode of the solution. If $\xi = 0$, then (1.3) is decomposed in two one-dimensional problems

$$\begin{cases} \overline{\rho} w_{\alpha t} - \mu \dfrac{d^2 w_\alpha}{dy_3^2} = f_{1\alpha}(y_3, t), \quad y_3 \in I^{\pm}, \\ [w_\alpha] = 0, \quad [\mu \dfrac{dw_\alpha}{dy_3}] = b_\alpha(t), \quad y_3 = 0, \\ w_\alpha|_{y_3 = \pm d_0} = 0, \\ w_\alpha|_{t=0} = 0 \end{cases} \tag{2.22}$$

with $\alpha = 1, 2$, $I^{\pm} = \{\pm y_3 \in (0, d_0)\}$ and

$$\begin{cases} \overline{\rho} w_{3t} - (2\mu + \mu_1) \dfrac{d^2 w_3}{dy_3^2} + p'(\overline{\rho}) \dfrac{d\vartheta}{dy_3} = f_3(y_3, t), \\ \vartheta_t + \overline{\rho} \dfrac{dw_3}{dy_3} = h(y_3, t), \quad y_3 \in I^{\pm}, \\ [w_3] = 0, \quad [-p'(\overline{\rho})\vartheta + (2\mu + \mu_1) \dfrac{dw_3}{dy_3}] = b_3(t), \quad y_3 = 0, \\ w_3|_{y_3 = \pm d_0} = 0, \\ w_3|_{t=0} = 0, \quad \vartheta|_{t_0=0} = 0 \end{cases} \tag{2.23}$$

(we recall that by our initial assumptions, $\boldsymbol{w}$ and θ vanish for $|y_3| \geq d_0$.) The parabolic problem (2.22) is easily studied, and we restrict ourselves with the analysis of (2.23). We make the Laplace transform, eliminate ϑ and replace $\nu_1^{\pm}$ with $\nu_1^{\pm}(s)$. This leads to

$$\begin{cases} s\widetilde{w}_3^{\pm} - (2\nu^{\pm} + \nu_1^{\pm}(s)) \dfrac{d^2 \widetilde{w}_3^{\pm}}{dy_3^2} = \dfrac{1}{\overline{\rho}^{\pm}} (\widetilde{f}_3^{\pm} - \dfrac{p'(\overline{\rho}^{\pm})}{s} \dfrac{d\widetilde{h}^{\pm}}{dy_3}), \quad y_3 \in I^{\pm}, \\ [\widetilde{w}_3] = 0, \quad [(2\mu + \mu_1(s)) \dfrac{d\widetilde{w}_3}{dy_3}] = \widetilde{b}_3 + [\dfrac{\overline{\rho} p'(\overline{\rho})}{s} \widetilde{h}], \quad y_3 = 0, \\ \widetilde{w}_3^{\pm}(\pm d) = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} R^{\pm}(s) \widetilde{w}_3^{\pm} - \dfrac{d^2 \widetilde{w}_3^{\pm}}{dy_3^2} = \dfrac{1}{\overline{\rho}^{\pm}} (\dfrac{s\widetilde{f}_3^{\pm}}{a^{\pm}s + b^{\pm}} - \dfrac{p'(\overline{\rho}^{\pm})}{a^{\pm}s + b^{\pm}} \dfrac{d\widetilde{h}^{\pm}}{dy_3}) \equiv \widetilde{g}^{\pm}, \quad y_3 \in I^{\pm}, \\ [\widetilde{w}_3] = 0, \quad [\lambda \dfrac{d\widetilde{w}_3}{dy_3}] = \dfrac{s}{a_0 s + b_0} \widetilde{b}_3 - [\dfrac{\overline{\rho}^{\pm} p'(\overline{\rho}^{\pm})}{a_0 s + b_0} \widetilde{h}] \equiv \widetilde{e}, \quad y_3 = 0, \\ \widetilde{w}_3^{\pm}(\pm d_0) = 0, \end{cases} \tag{2.24}$$

where

$$R^{\pm}(s) = \frac{\overline{\rho}^{\pm} s^2}{a_0^{\pm} s + b_0^{\pm}}, \quad \lambda^{\pm} = \overline{\rho}^{\pm} \frac{a_0^{\pm} s + b_0^{\pm}}{a_0 s + b_0}$$

and a_0, b_0 are some fixed positive numbers, for instance, $a_0 = max(a^+, a^-)$, $b_0 = max(b^+, b^-)$.

We introduce $\widetilde{u}_1^{\pm}$ as the solution of

$$R^{\pm}\widetilde{u}_1^{\pm} - \frac{d^2\widetilde{u}_1^{\pm}}{dy_3^2} = g^{\pm}, \quad y_3 \in I^{\pm}, \quad \widetilde{u}_1 = 0, \quad y_3 = 0, \pm d_0.$$

To estimate $\widetilde{u}_1^{\pm}$, we expand $g^{\pm}$ in the Fourier series in $\sin \frac{k\pi y_3}{d_0}$, $k = 1, 2, \ldots$, Then the Fourier coefficients $\widehat{u}_1^{\pm}$ of $\widetilde{u}_1^{\pm}$ are defined by

$$\widehat{u}_1^{\pm}(\xi, s) = \frac{\widehat{g}^{\pm}(\xi, s)}{R^{\pm}(s) + |\xi|^2}, \quad \xi = k\pi/d_0, \quad k = 1, 2\ldots. \tag{2.25}$$

If d_0 is small, then, by the Marcinkiewicz theorem,

$$\|e^{\beta t}\widetilde{u}_1^{\pm}\|_{W_p^{2,1}(I_\infty^{\pm})} \leq c\|e^{\beta t}g^{\pm}\|_{L_p(I_\infty^{\pm})}, \quad I_\infty^{\pm} = I^{\pm} \times \mathbb{R}_+. \tag{2.26}$$

The difference $\widetilde{u}_2 = \widetilde{w}_3 - \widetilde{u}_1$ satisfies

$$\begin{cases} R^{\pm}\widetilde{u}_2^{\pm} - \dfrac{d^2\widetilde{u}_2^{\pm}}{dy_3^2} = 0, \quad y_3 \in I^{\pm}, \\ [\widetilde{u}_2] = 0, \quad [\lambda \dfrac{d\widetilde{u}_2}{dy_3}] = -[\lambda \dfrac{d\widetilde{u}_1}{dy_3}] \equiv \widetilde{A}(s), \quad y_3 = 0, \\ \widetilde{u}_2^{\pm} = 0, \quad y = \pm d. \end{cases} \tag{2.27}$$

We need to compute the solution of this problem. It is given by

$$u_2^+(y_3) = C^+(e^{r^+(d_0-y_3)} - e^{-r^+(d_0-y_3)}), \quad y_3 \in I^+,$$
$$u_2^-(y_3) = C^-(e^{r^-(d_0+y_3)} - e^{-r^-(d_0+y_3)}), \quad y_3 \in I^-,$$

where

$$r^{\pm} = \frac{s}{\sqrt{a^{\pm}s + b^{\pm}}}.$$

From the jump conditions on the interface $y_3 = 0$ it follows that

$$\mathfrak{D}(C^+, C^-)^T = (\widetilde{A}, 0)^T,$$

where

$$\mathfrak{D} = \begin{pmatrix} -\lambda^+ r^+(e^{r^+d_0} + e^{-r^+d_0}) & -\lambda^- r^-(e^{r^-d_0} + e^{-r^-d_0}) \\ e^{d_0r^+} - e^{-d_0r^+} & -(e^{d_0r^-} - e^{-d_0r^-}) \end{pmatrix},$$

so that

$$D = det\mathfrak{D} = \lambda^+ r^+(e^{d_0r^+} + e^{-d_0r^+})(e^{d_0r^-} - e^{-d_0r^-}) + \lambda^- r^-(e^{d_0r^-} + e^{-d_0r^-})(e^{d_0r^+} - e^{-d_0r^+}).$$

Hence

$$C^+ = -\frac{\widetilde{A}}{D}(e^{d_0r^-} - e^{-d_0r^-}), \quad C^- = -\frac{\widetilde{A}}{D}(e^{d_0r^+} - e^{-d_0r^+}).$$

We also compute

$$\widetilde{U}^{\pm} = \frac{d\widetilde{u}_2^{\pm}}{dy_3}\Big|_{y_3=0} = \mp r^{\pm}C^{\pm}(e^{r^{\pm}d_0} + e^{-r^{\pm}d_0}).$$

These functions are connected with $\widetilde{A}$ by

$$\widetilde{U}^+(s) = \frac{r^+\widetilde{A}}{D}(e^{d_0r^+} + e^{-d_0r^+})(e^{d_0r^-} - e^{-d_0r^-})$$
$$= \frac{r^+\widetilde{A}(1+e^{-2d_0r^+})(1-e^{-2d_0r^-})}{\lambda^+r^+(1+e^{-2d_0r^+})(1-e^{-2d_0r^-}) + \lambda^-r^-(1+e^{-2d_0r^-})(1-e^{-2d_0r^+})} \equiv \mathcal{W}^+\widetilde{A}(s),$$
$$\widetilde{U}^-(s) = -\frac{r^-\widetilde{A}(1+e^{-2d_0r^-})(1-e^{-2d_0r^+})}{\lambda^+r^+(1+e^{-2d_0r^+})(1-e^{-2d_0r^-}) + \lambda^-r^-(1+e^{-2d_0r^-})(1-e^{-2d_0r^+})} \equiv \mathcal{W}^-\widetilde{A}(s). \tag{2.28}$$

It is easily verified that the expressions $\mathcal{W}^\pm(s)$ are L_p-multipliers, hence

$$\|e^{\beta t}U^\pm\|_{W_p^{1/2-1/2p}(\mathbb{R}_+)} \le c\|e^{\beta t}A\|_{W_p^{1/2-1/2p}(\mathbb{R}_+)}. \tag{2.29}$$

Now we consider (2.27) as the union of two problems

$$\begin{cases} R^\pm\widetilde{u}_2^\pm - \dfrac{d^2\widetilde{u}_2^\pm}{dy_3^2} = 0, \quad y_3 \in I^\pm, \\ \dfrac{d\widetilde{u}_2^\pm}{dy_3}|_{y_3=0} = \widetilde{U}^\pm, \\ \widetilde{u}_2^\pm = 0, \quad y = \pm d. \end{cases} \tag{2.30}$$

The solution of (2.30) can be estimated in the same way as $\widetilde{u}_1$ above. We have $\widetilde{u}_2^\pm = \widetilde{V}^\pm + \widetilde{W}^\pm$, where

$$\frac{d\widetilde{V}^\pm}{dy_3}|_{y_3=0} = \widetilde{U}^\pm, \quad \widetilde{V}^\pm|_{y_3=0} = \widetilde{V}^\pm|_{y_3=\pm d_0} = 0,$$
$$\|e^{\beta t}V^\pm\|_{W_p^{2,1}(I_\infty^\pm)} \le c\|e^{\beta t}A\|_{W_p^{1/2-1/2p}(\mathbb{R}_+)}, \quad I_\infty^\pm = I^\pm \times (0,\infty), \tag{2.31}$$
$$\begin{cases} R^\pm(s)\widetilde{W}^\pm - \dfrac{d^2\widetilde{W}^\pm}{dy_3^2} = -R^\pm(s)\widetilde{V}^\pm + \dfrac{d^2\widetilde{V}^\pm}{dy_3^2} = \widetilde{F}^\pm, \quad y_3 \in I^\pm, \\ \dfrac{d\widetilde{W}^\pm}{dy_3}|_{y_3=0} = 0, \quad \widetilde{W}^\pm|_{y_3=d_0} = 0. \end{cases}$$

We extend $\widetilde{F}^\pm$ as an even function of y_3 into a symmetric interval $I^\mp$ and apply the Laplace transform. It is clear that $\widehat{W}^\pm$ is expressed by the formula similar to (2.25):

$$\widehat{W}^\pm(\xi, s) = \frac{\widehat{F}(\xi, s)}{R^\pm(s) + |\xi|^2}. \tag{2.32}$$

Applying the Marcinkiewicz theorem and making use of (2.31), we obtain

$$\sum_\pm \|e^{\beta t}\widetilde{u}_2^\pm\|_{W_p^{2,1}(I_\infty^\pm)} \le c\sum_\pm \|e^{\beta t}V^\pm\|_{W_p^{2,1}(I_\infty^\pm)} \le c\|e^{\beta t}A\|_{W_p^{1/2-1/2p}(\mathbb{R}_+)},$$

which completes the proof of

$$\sum_\pm(\|e^{\beta t}w_3\|_{W_p^{2,1}(I_\infty^\pm)} + \|e^{\beta t}\vartheta\|_{W_p^{1,0}(I_\infty^\pm)} + \|e^{\beta t}\vartheta_t\|_{W_p^{0,1}(I_\infty^\pm)})$$
$$\le c\Big(\sum_\pm(\|e^{\beta t}f_{13}\|_{L_p(I_\infty^\pm)} + \|e^{\beta t}h\|_{W_p^{0,1}(I_\infty^\pm)}) + \|e^{\beta t}\boldsymbol{b}\|_{W_p^{1/2-1/(2p)}(\mathbb{R}_+)}\Big). \tag{2.33}$$

(the estimates of ϑ are obtained from the equations (2.23)).

The solution of (1.24) satisfies similar inequalities.

Combining (2.33) with (2.9) and (2.21) we obtain inequality (2.1).

Next, we turn to the problem (1.31) with $\boldsymbol{f}=0$, $h=0$ and with $\boldsymbol{a}$ vanishing for $t=0$ (if $p\geq 3/2$). Assuming as above that $|\xi|\geq \pi/d_0$, we write (1.37) as

$$\begin{gathered}\widetilde{\boldsymbol{w}}(\xi,s.y_3)=\begin{pmatrix} r\widetilde{a}_1\\ r\widetilde{a}_2\\ r\widetilde{a}_3\end{pmatrix} r^{-1}e^{-ry_3}+h_3(r_{11}-r)\begin{pmatrix} i\xi_1\\ i\xi_2\\ -r_{11}\end{pmatrix}\frac{e^{-r_{11}y_3}-e^{-ry_3}}{r_{11}-r}\qquad y_3>0,\\ h_3(r_{11}-r)=\frac{a_1s+b}{as+b}\frac{\widetilde{A}-r\widetilde{a}_3}{r_{11}+\varkappa(s)r},\qquad \varkappa(s)=\frac{\nu s}{as+b}.\end{gathered}\tag{2.34}$$

It is easily seen that the assumptions of Proposition 1 are satisfied in the formula for $\mathcal{F}D_y^jD_t^k\boldsymbol{w}$, $2|j|+k\leq 2$, the role of $\widetilde{d}_i$ being played by $r\widetilde{a}_i$. Hence there holds

$$\|e^{\beta t}\boldsymbol{w}\|_{W_p^{2,1}(Q_\infty^\pm)}\leq c\|e^{\beta t}F^{-1}r\widetilde{\boldsymbol{a}}\|_{W_p^{1-1/p,1/2-1/2p}(Q'_\infty)}\leq c\|e^{\beta t}\boldsymbol{a}\|_{W_p^{2-1/p,1-/2p}(Q'_\infty)}.\tag{2.35}$$

We notice that this inequality justifies (2.7).

As for the non-homogeneous problem (1.31), it reduces to a homogeneous one exactly as (1.3); detail are omitted. For the solution of (1.31) we obtain the estimate

$$\begin{gathered}\|e^{\beta t}\boldsymbol{w}\|_{W_p^{2,1}(Q_\infty^\pm)}+\|e^{\beta t}\sigma_t\|_{W_p^{1,0}(Q_\infty^\pm)}+\|e^{\beta t}\sigma\|_{W_p^{1,0}(Q_\infty^\pm)}\\ \leq c(\|e^{\beta t}a\|_{W_p^{2-1/p,1-/2p}(Q'_\infty)}+\|e^{\beta t}\boldsymbol{f}\|_{L_p(Q_\infty^\pm)}+\|e^{\beta t}h\|_{W_p^{1,0}(Q_\infty^\pm)}).\end{gathered}\tag{2.36}$$

If $\xi=0$, then (1.32) reduces to two one-dimensional problems

$$\begin{cases}\overline{\rho}s\widetilde{w}_\alpha-\mu\dfrac{d^2\widetilde{w}_\alpha}{dy_3^2}=\widetilde{f}_\alpha(y_3,t),\quad y_3\in I^+,\\ \widetilde{w}_\alpha|_{y_3=0}=\widetilde{a}_\alpha,\quad \widetilde{w}_\alpha|_{y_3=d_0}=0,\end{cases}\tag{2.37}$$

$\alpha=1,2$, and

$$\begin{cases}\overline{\rho}sw_{3t}-(2\mu+\mu_1)\dfrac{d^2\widetilde{w}_3}{dy_3^2}+p'(\overline{\rho})\dfrac{\widetilde{\vartheta}}{dy_3}=\widetilde{f}_3(y_3,t),\\ \widetilde{\vartheta}_t+\overline{\rho}\dfrac{d\widetilde{w}_3}{dy_3}=h,\quad y_3\in I^+,\\ \widetilde{w}_3|_{y_3=0}=\widetilde{a}_3,\quad \widetilde{w}_3|_{y_3=d_0}=0.\end{cases}\tag{2.38}$$

They are treated as (2.24), even much easier.

Finally, we mention the interior estimates of the solution of (1.3). The corresponding model problem is completely analogous to (1.24):

$$\begin{cases}\overline{\rho}\boldsymbol{w}_t-\nabla\cdot T(\boldsymbol{w})+p'(\overline{\rho})\nabla\vartheta=\boldsymbol{f},\\ \vartheta_t+\overline{\rho}\nabla\cdot\boldsymbol{w}=h,\quad (y,t)\in Q_\infty,\\ \boldsymbol{w}(y,0)=0,\quad \vartheta(y,0)=0,\quad y\in Q.\end{cases}\tag{2.39}$$

Repeating the above arguments, we obtain

$$\begin{aligned}&\|e^{\beta t}\boldsymbol{w}\|_{W_p^{2,1}(Q_\infty)}+\|e^{\beta t}\vartheta\|_{W_p^{1,0}(Q_\infty)}+\|e^{\beta t}\vartheta_t\|_{W_p^{1,0}(Q_\infty)}\\&\leq c(\|e^{\beta t}\boldsymbol{f}\|_{L_p(Q_\infty)}+\|e^{\beta t}h\|_{W_p^{1,0}(Q_\infty)}).\end{aligned}\tag{2.40}$$

We have considered model problems only with zero initial data. This does not restrict generality, because problem (1.1) can be reduced to the problem with homogeneous initial data by constructing, after passage to the Lagrangian coordinates, auxiliary functions $\boldsymbol{U}^\pm(y,t),\Theta^\pm(y,t)$ such that

$$\begin{aligned}&\boldsymbol{U}^\pm(y,0)=\boldsymbol{v}_0^\pm(y),\quad \Theta^\pm(y,0)=\theta_0^\pm(y),\quad [\boldsymbol{U}]|_{y\in\Gamma_0}=0,\\&\boldsymbol{U}(y,t)=0,\quad \Theta(y,t)=0\quad\text{for}\quad t>2,\\&\sum_\pm\|\boldsymbol{U}^\pm\|_{W_p^{2,1}(\Omega_0^\pm\times(0,2))}\leq c\sum_\pm\|\boldsymbol{u}_0^\pm\|_{W_p^{2-2/p}(\Omega_0^\pm)},\\&\|\Theta^\pm\|_{W_p^{1,0}(\Omega_0^\pm\times(0,2))}+\|\Theta_t^\pm\|_{W_p^{1,0}(\Omega_0^\pm\times(0,2))}\leq c\|\theta_0^\pm\|_{W_p^1(\Omega_0^\pm)}.\end{aligned}\tag{2.41}$$

The existence of $\boldsymbol{U}^\pm$ with these properties follows from the trace theorem for the anisotropic Sobolev spaces, and $\Theta^\pm$ can be taken in the form $\Theta^\pm(y,t)=\theta_0^\pm(y)\zeta(t)$, where $\zeta(t)$ is a smooth function vanishing for $t>2$ and equal to one for small t.

Inequalities (2.2), (2.37), (2.39) in combination with localization procedure yield the following estimate for the linear problem (1.2):

$$\begin{aligned}&\sum_\pm\|e^{\beta t}\boldsymbol{v}^\pm\|_{W_p^{2,1}(\Omega_0^\pm\times(0,\infty))}+\|e^{\beta t}\theta^\pm\|_{W_p^{1,0}(\Omega_0^\pm\times(0,\infty))}+\|e^{\beta t}\theta_t^\pm\|_{W_p^{1,0}(\Omega_0^\pm\times(0,\infty))}\\&\leq c(\sum_\pm(\|e^{\beta t}\boldsymbol{f}^\pm\|_{L_p(\Omega_0^\pm\times(0,\infty))}+\|e^{\beta t}h^\pm\|_{W_p^{0,1}(\Omega_0^\pm\times(0,\infty))}+\|\boldsymbol{v}_0^\pm\|_{W_p^{2-2/p}(\Omega_0^\pm)}+\|\theta_0^\pm\|_{W_p^{1,0}(\Omega_0^\pm)})\\&+\|e^{\beta t}\boldsymbol{v}\|_{L_p(\Omega_0\times(0,\infty))}+\|e^{\beta t}\theta\|_{L_p(\Omega_0\times(0,\infty))})\end{aligned}\tag{2.42}$$

The estimate of weighted L_p-norms of $\boldsymbol{v}$ and θ can be obtained on the basis of the resolvent estimate of the operator corresponding to the problem (1.2) and of the standard results of the semi-group theory; this goes beyond the frames of the present paper.

Remark. We have studied model problems (1.3) and (1.31) in the class of functions decaying exponentially as $t\to\infty$. Most often, such problems are treated by applying the integral Fourier-Laplace transformation. It is easily seen that the above arguments are applicable also in this case, but the exponential weight will be with $\beta\leq 0$, which suffices for estimating solutions of (1.2) in a finite time interval.

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